Year 13 Mathematics

EAS 3.7

Integration

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Contents

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Integration 3.7

This achievement standard involves applying integration methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply integration methods in solving problems.	• Apply integration methods, using relational thinking, in solving problems.	• Apply integration methods, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives:
  - choose and apply a variety of integration and anti-differentiation techniques to functions and relations using both analytical and numerical methods
  - form differential equations and interpret the solutions

  in the Mathematics strand of the Mathematics and Statistics Learning Area.

- Apply integration methods in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.

- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and relating findings to a context, or communicating thinking using appropriate mathematical statements.

- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and using correct mathematical statements, or communicating mathematical insight.

- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

- Methods include a selection from those related to:
  - integrating power, polynomial, exponential (base e only), trigonometric, and rational functions
  - reverse chain rule, trigonometric formulae
  - rates of change problems
  - areas under or between graphs of functions, by integration
  - finding areas using numerical methods, e.g. the rectangle or trapezium rule
  - differential equations of the forms $y' = f(x)$ or $y'' = f(x)$ for the above functions or situations where the variables are separable (e.g. $y' = ky$) in applications such as growth and decay, inflation, Newton's Law of Cooling and similar situations.

Integration of Polynomials

Integration of Polynomials

Integration is the reverse process of differentiation and is also called antidifferentiation.

To integrate a polynomial we do so term-by-term using the rule:

Increase the power of the term by one and then divide by the new power.

We can write this as

$$\int kx^n\,dx = \frac{k}{n+1}x^{n+1} + C$$

1. Increase the power by one.
2. Divide by the new power.
3. Add a new constant C.

To integrate a polynomial we must ensure each term of the expression is in the form kx^n.

The symbol $\int$ is used to denote the integral and dx tells us which variable we are finding the integral of.

Example

Integrate the expression

$$I = \int 12x^3 - 6x + 3 - 6\sqrt{x} + \frac{3}{x^2}\,dx$$

We begin by rewriting all the terms in the expression in the form kx^n.

$$I = \int 12x^3 - 6x + 3 - 6x^{1/2} + 3x^{-2}\,dx$$

Integrating term-by-term

$12x^3$ becomes $\frac{12x^{3+1}}{4} = 3x^4$

$-6x$ becomes $\frac{-6x^{1+1}}{2} = -3x^2$

3 becomes $\frac{3x^{0+1}}{1} = 3x$

$-6x^{1/2}$ becomes $\frac{-6x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{-6x^{3/2}}{\frac{3}{2}} = -4x^{3/2}$

and $3x^{-2}$ becomes $\frac{3x^{-2+1}}{-1} = -3x^{-1}$

so

$$I = \int 12x^3 - 6x + 3 - 6x^{1/2} + 3x^{-2}\,dx$$

becomes

$$I = 3x^4 - 3x^2 + 3x - 4x^{3/2} - 3x^{-1} + C$$

$$= 3x^4 - 3x^2 + 3x - 4\sqrt{x^3} - \frac{3}{x} + C$$

Polynomial

A polynomial is a mathematical expression comprising a sum of terms where each term includes a variable raised to a power and multiplied by a coefficient, e.g. $2x^3 + 3x^2 - 4x + 1$.

Constant of Integration

When we differentiate any function any constants (e.g. 5) differentiates to 0. Therefore, when we are doing the reverse and integrating it is not possible to identify a constant in the resulting integral. Demonstrating this

If $f(x) = 4x^6 - 3x + 5$

$f'(x) = 24x^5 - 3$

If we now integrate $f'(x)$ to recover $f(x)$ we no longer have the information needed to recover the constant 5.

To get around this problem, when integrating an expression, we always add a constant C at the end of the integral.

Integration of x^{-1}

The given integration rule cannot be used to integrate x^{-1}, because increasing the power of x by one and dividing by the new power would result in $\frac{1}{0}$ which is undefined. We need to use another approach to integrate $\frac{1}{x}$, **see Page 11.**

Example

Find $\int A(x+1)^2\,dx$

We begin by putting the constant, A in front of the integral sign before we integrate.

$$= A\int (x+1)^2\,dx$$

$$= A\int (x+1)(x+1)\,dx$$

$$= A\int x^2 + 2x + 1\,dx$$

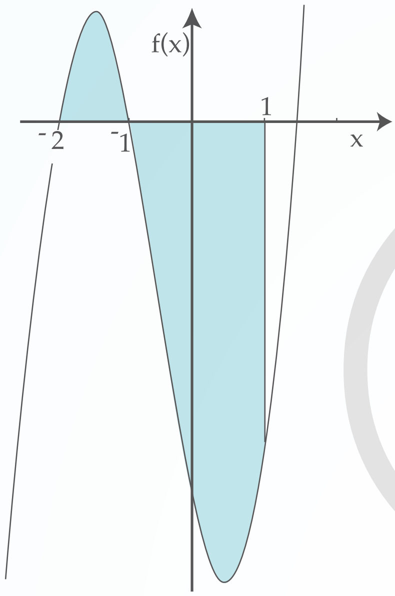
$$= A\left(\frac{x^3}{3} + \frac{2x^2}{2} + x\right) + C$$

$$= A\left(\frac{x^3}{3} + x^2 + x\right) + C$$

For the example above we could have expressed the constant as AC, because A is a multiplier for the entire integral, but as the constant of integration is an unknown constant, AC is also an unknown constant, so we just represent it with C.

Example

Find the area between the curve $f(x) = 2x^3 + 3x^2 - 5x - 6$ and the x axis from $x = {}^-2$ to $x = 1$.



Solution

Area = Area above + Area below

$$\text{Area} = \int_{-2}^{-1} 2x^3 + 3x^2 - 5x - 6\,dx + \left|\int_{-1}^{1} 2x^3 + 3x^2 - 5x - 6\,dx\right|$$

$$\text{Area} = \left[\frac{2}{4}x^4 + x^3 - \frac{5}{2}x^2 - 6x\right]_{-2}^{-1} + \left[\frac{2}{4}x^4 + x^3 - \frac{5}{2}x^2 - 6x\right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - 1 - \frac{5}{2} + 6\right) - \left(\frac{16}{2} - 8 - \frac{20}{2} + 12\right) + \left(\frac{1}{2} + 1 - \frac{5}{2} - 6\right) - \left(\frac{1}{2} - 1 - \frac{5}{2} + 6\right)$$

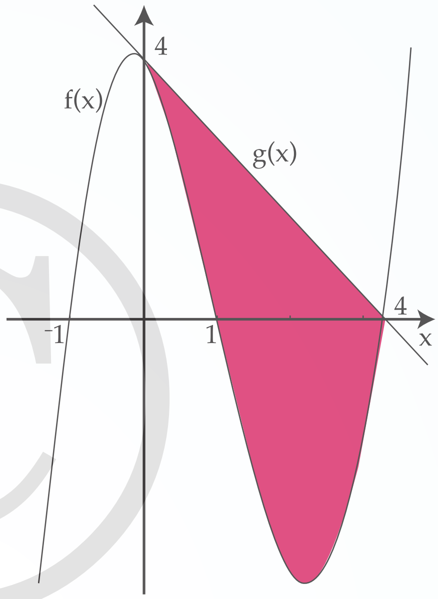
$= 3 - 2 + |{}^-7 - 3|$

$= 1 + 10$

$= 11$ units^2

Example

Find the area between the curve $f(x) = x^3 - 4x^2 - x + 4$ and the line $g(x) = 4 - x$ from $x = 0$ to $x = 4$.



Solution

Although the curve is above and below the x axes we are only interested in the enclosed area.

If we let $D(x) = g(x) - f(x)$

then as $g(x)$ is always above $f(x)$ from $x = 0$ to $x = 4$ then $D(x)$ is positive in this region.

Therefore we can integrate it to find the area enclosed.

$$\text{Area} = \int_0^4 D(x)\,dx$$

$$= \int_0^4 g(x) - f(x)\,dx$$

$$= \int_0^4 (4 - x) - (x^3 - 4x^2 - x + 4)\,dx$$

$$= \int_0^4 {}^-x^3 + 4x^2\,dx$$

$$= \left[\frac{{}^-x^4}{4} + \frac{4}{3}x^3\right]_0^4$$

$$= 21\frac{1}{3} \text{ units}^2$$

It is important we set D(x) equal to the higher expression (g(x) minus the lower expression f(x)) over the range otherwise the integral would be negative.

Rates of Change

Rates of Change – Kinematics

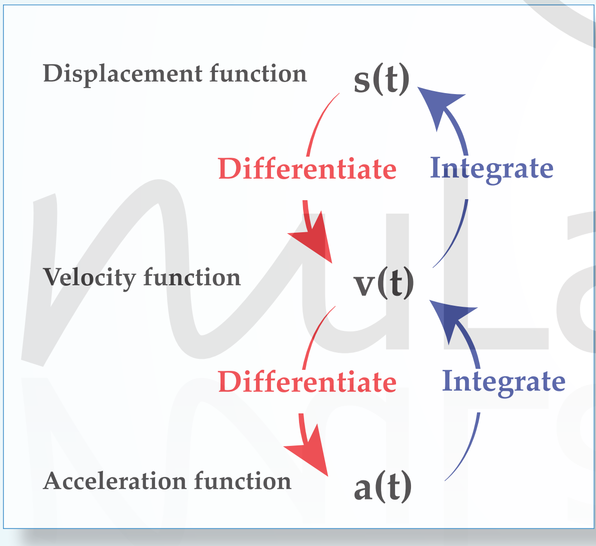
Application problems that involve functions of time, i.e. displacement, velocity and acceleration can be solved using calculus techniques.

Since differentiation gives us the instantaneous rate of change we can use it to find the rate of change of different expressions.

Displacement is the distance in a particular direction. The rate of change of displacement is **velocity** and the rate of change of velocity is **acceleration**.

If we want to form a velocity function from a displacement function we differentiate. If we want to form a displacement function from a velocity function we integrate.

The same applies to velocity and acceleration.



For example, if we have the velocity function of a ball thrown vertically upwards

$v(t) = 24 - 10t$ m/s

The displacement function for this ball is

$s(t) = \int v(t)\, dt$

$s(t) = \int 24 - 10t\, dt$

$s(t) = 24t - 5t^2 + C$ m

where C is the displacement at time $t = 0$.

Similarly, if we had the same velocity function and we wanted the acceleration function we differentiate

$v(t) = 24 - 10t$

$a(t) = {}^{-}10$ m/s^2

Extra Notes

Velocity and Speed

The correct term for the rate at which displacement changes is velocity. Speed is just a measure of how quickly distance is changing and does not take into account the direction.

Units for Velocity and Acceleration

If the units for displacement are metres and time is measured in seconds, then the units for velocity are metres per second (m/s) and the units for acceleration are metres per second squared (m/s²).

Look out for these key words

At 'rest' means velocity is zero ($v(t) = 0$).

Constant speed means no acceleration, $a(t) = 0$.

Maximum (or minimum) displacement means the object has momentarily stopped and is about to head back. The velocity is always zero ($v(t) = 0$) when the distance is a maximum (or minimum).

Initial position or velocity is when $t = 0$.

For displacement:

$s(0)$ gives the initial displacement of the object ($t = 0$).

$s = 0$ is used to solve for the time(s) t when the object is at the reference point.

For velocity:

$v(0)$ gives the initial velocity of the object ($t = 0$).

$v = 0$ is used to solve for the time t when the object is momentarily at rest.

$v > 0$ the object is travelling away from the start.

$v < 0$ the object is travelling back to (towards) the start.

For acceleration:

$a(0)$ gives the object's initial acceleration.

$a = 0$ is used to solve for the time t when the object is not accelerating (speed or velocity is constant).

$a < 0$ the object is slowing down.

$a > 0$ the object is speeding up.

Simpson's Rule cont...

Generalising the rule to cover n columns we get

$$\int_a^b f(x)\,dx = \frac{1}{3}h[y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

where $h = \dfrac{b-a}{n}$ and $y_r = f(x_r)$ and n is even.

n must be even as the rule requires pairs of columns.

Example

Using Simpson's rule, find the area under the function $f(x) = x(x + 2)$ between $x = 0$ and $x = 4$ using eight subintervals.

$$h = \frac{b-a}{n} = \frac{4-0}{8} = 0.5$$

This is the same example as in the Trapezium rule.

We calculate the required values in the table below. The table helps make sure we keep the first + last, even columns and odd columns separate.

n	x_n	$y_0 + y_n$	y_{Odd}	y_{Even}
0	0	0		
1	0.5		1.25	
2	1.0			3.00
3	1.5		5.25	
4	2.0			8.00
5	2.5		11.25	
6	3.0			15.00
7	3.5		19.25	
8	4.0	24.00		
Sums of y		24.00	37.00	26.00

$$\text{Area} = \frac{h}{3}(y_0 + y_6 + 4y_{Odd} + 2y_{Even})$$

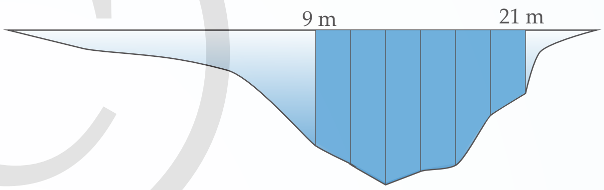
$$= \frac{0.5}{3}(24 + 4 \times 37 + 2 \times 26)$$

$$= 37.33 \text{ units}^2$$

Example

We need to know the cross-sectional area of a boat channel in a river. The table has depth measurements from the shallow side of the river. The river flows down the boat channel at 4.3 m/s. Use Simpson's rule to calculate the cross-sectional area and volume of water per second.

x	9	11	13	15	17	19	21
Depth (y)	13.3	14.5	17.2	15.3	14.8	12.7	9.7



$$h = \frac{b-a}{n} = \frac{21-9}{6} = 2.0 \text{ m}$$

With Simpson's rule n must be even.

Watch that you have a zigzag pattern in your table.

n	x_n	$y_0 + y_n$	y_{Odd}	y_{Even}
0	9	13.3		
1	11		14.5	
2	13			17.2
3	15		15.3	
4	17			14.8
5	19		12.7	
6	21	9.7		
Sums of y		23.0	42.50	32.0

$$\text{Area} = \frac{h}{3}(y_0 + y_6 + 4y_{Odd} + 2y_{Even})$$

$$= \frac{2}{3}(23.0 + 4 \times 42.5 + 2 \times 32.0)$$

$$= 171.3 \text{ m}^2$$

$$\text{Volume} = \text{area} \times \text{rate of flow}$$

$$= 736.6 \text{ m}^3/\text{s}$$

Example

If the rate at which an investment in art appreciates depends upon the value of the piece of art

$$\frac{dP}{dt} = iP$$

find the annual rate of appreciation of an artwork which was purchased for $12 500 and was valued at $14 300 five years later.

Solution

For $\frac{dP}{dt} = iP$

the general solution is

$$P = P_0 e^{it}$$

Since $P_0 = 12\,500$ when $t = 0$

$$P = 12\,500e^{it}$$

When $t = 5$, $P = 14\,300$ so

$$14\,300 = 12\,500\, e^{5i}$$

Solving for i by dividing through by 12 500

$$e^{5i} = 1.144$$

Taking logs of both sides

$$\ln(e^{5i}) = \ln(1.144)$$

Since $\ln(e^{5i}) = 5i$ and $\ln(1.144) = 0.134\,53$

$$5i = 0.134\,53$$

$$i = 0.0269 \qquad \text{(3 sf)}$$

$$= 2.7\ \%\ \text{pa}$$

Example

Assuming the rate of inflation is constant and it appreciates according to the differential equation,

$$\frac{dP}{dt} = iP$$

find the value (to 3 sf) of a house in January 2017 if it was valued at $240 000 in January 1997 and $620 000 in July 2008.

Solution

For $\frac{dP}{dt} = iP$

the general solution is

$$P = P_0 e^{it}$$

Let January 1997 be $t = 0$

$$240\,000 = P_0 e^0$$

$$P_0 = 240\,000$$

so P_0 becomes the initial value at $t = 0$.

In July 2008, $t = 11.5$ so

$$620\,000 = 240\,000\, e^{11.5i}$$

$$e^{11.5i} = \frac{620\,000}{240\,000}$$

Solving for i by taking logs of both sides

$$\ln(e^{11.5i}) = \ln(2.5833)$$

$$11.5i = 0.949\,081$$

$$i = 0.082\,529\ (8.25\%)$$

So in January 2017 when $t = 20$

$$P = 240\,000e^{0.082\,529 \times 20}$$

$$P = \$1\,250\,000 \text{ (3 sf)}$$

Page 14 cont...

70. $\frac{Ax^2}{2} + B\ln|x| + \frac{D}{x} + C$

71. $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 5\ln|x| + C$

72. $x^5 + 2\ln|x| - \frac{3}{x} + C$

73. $\frac{1}{5}x^5 + \frac{1}{6}x^2 - \frac{5}{3}\ln|x| + C$

74. $\ln|x| + \frac{B}{Ax} + C$

Page 16

75. $3x - 11\ln|x+1| + C$

76. $3x + 9\ln|x-2| + C$

77. $2x - \frac{7}{4}\ln|4x+5| + C$

78. $3x + 18\ln|x-6| + C$

79. $3x + \frac{1}{2}\ln|2x-1| + C$

80. ${}^{-}6x - 6\ln|1-x| + C$

81. $2x + \frac{1}{4}\ln|4x+3| + C$

82. $\frac{x}{2} + \frac{1}{2}\ln|2x-2| + C$

Page 17

83. $2x - \ln|x-1| + C$

84. ${}^{-}3x + 13\ln|x+4| + C$

85. $x - 4\ln|2x+3| + C$

86. $3x + 4\ln|3-2x| + C$

87. $4x + 8\ln|x-2| + C$

88. $\frac{x}{3} - \frac{2}{9}\ln|3x+2| + C$

89. ${}^{-}7x - 32\ln|x-5| + C$

90. $2x - \ln|2-x| + C$

Page 19

91. $\ln|6x+3| + C$

92. $3\ln|x-2| + C$

93. $2\ln|x^2+1| + C$

94. $\frac{A}{3}\ln|3x-1| + C$

95. $\ln|x^2-3x+1| + C$

96. $2\ln|x^2+x-2| + C$

97. $2\ln|x^2-1| + C$

98. $\ln|e^{2x}-5| + C$

Page 20

99. $\ln|\tan 2x| + C$

100. $\ln|\cos 3x| + C$

101. $2\ln|\text{cosec } 5x| + C$

102. $2\ln|7 - \sin 4x| + C$

103. $\ln|\ln|x|| + C$

104. $\ln|e^{3x}+7| + C$

105. $2\ln|e^{x^2}-3| + C$

106. ${}^{-}\ln\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$

107. $\ln|\cos x + \sin x| + C$

108. $\frac{1}{2}\ln|e^{2x}+2x| + C$

Page 22

109. $\int 4(\sin 6x + \sin 4x)\,dx$

$= \frac{{}^{-}2}{3}\cos 6x - \cos 4x + C$

110. $\int 4(\sin 4x - \sin 2x)\,dx$

$= {}^{-}\cos 4x + 2\cos 2x + C$

111. $6\sin 2x - 2\sin 6x + C$

112. ${}^{-}0.4\cos 10x - 2\cos 2x + C$

113. $\frac{5}{8}\sin 8x + \frac{5}{2}\sin 2x + C$

114. $\frac{{}^{-}3}{22}\cos 11x + \frac{3}{2}\cos x + C$

Page 23

115. $4\tan x - 4x + C$

116. $\frac{1}{4}\sin 2x + \frac{1}{2}x + C$

117. $\sin 2x + 2x + C$

118. ${}^{-}20\cot x - 20x + C$

119. $\frac{1}{2}\sin 2x + C$ OR
$\sin x\cos x + C$

120. $2x - \sin 2x + C$ OR
${}^{-}2\sin x\cos x + 2x + C$

121. $6x + 3\sin\left(2x - \frac{\pi}{3}\right) + C$ Other forms of this answer possible.

122. ${}^{-}4\cos\left(2x + \frac{2\pi}{5}\right) + C$ Other forms of this answer possible

123. $2x^2 + 6x - \frac{3}{2}\sin 4x + C$

124. $6x - 4\sin 2x + \frac{1}{2}\sin 4x + C$

Page 25

125. $(2x+3)^6 + C$

126. $\frac{{}^{-}6}{(x-2)^5} + C$

127. $3(x-6)^4 + C$

128. $\frac{{}^{-}8}{(x+2)^3} + C$ OR
${}^{-}8(x+2)^{-3} + C$

Page 26

129. $5(x+3) - 9\ln|x+3| + C$

130. $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

131. $\frac{1}{2}(x^2+4)^6 + C$

132. $8\sqrt{x+2} + C$

133. $\frac{1}{7}(x+5)^7 - \frac{5}{3}(x+5)^6 + 5(x+5)^5 + C$

134. $\int (2u+5)u^5\ du$

$= \int 2u^6 + 5u^5\ du$

$= \frac{2}{7}(x-2)^7 + \frac{5}{6}(x-2)^6 + C$

135. $\ln|x-3| - \frac{3}{x-3} + C$

136. $\frac{{}^{-}1}{2(x^2+4x+5)^2} + C$

Page 27

137. $25(x+2)^{6/5} + C$

138. $\frac{1}{8}(x^2+5)^4 + C$

139. $\ln|\ln|x|| + C$

140. $\ln|e^x - 2| + C$

141. $\frac{1}{3}(2x-1)^{3/2} + (2x-1)^{1/2} + C$

$= \frac{1}{3}\sqrt{(2x-1)^3} + \sqrt{2x-1} + C$

142. $\frac{1}{3}\sqrt{(2x-1)^3} + \sqrt{2x-1} + C$

$= \frac{2}{3}(x+1)\sqrt{2x-1} + C$

143. $3e^{x^2} + C$

144. $3e^{x^2} + C$